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THE IMPACT OF CURVE SHORTENING ON THE WRITHE OF A CURVE

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ABSTRACT. This paper discusses the results of an undergraduate summer research project done at the University of Puget Sound during 2002 focusing on the curve shortening flow for space curves and its connections with the knot theory of the curve's associated Frenet ribbon. Several conjectures and further lines of research are suggested. Evolution equations for writhe are also derived. The reader is assumed to have basic knowledge of vector calculus.

1. INTRODUCTION

The curve shortening flow has previously been investigated by Gage [11], Hamilton [12], Grayson [14], Altschuler [4], and others, but until recently has focused on planar curves and surfaces. Recent work by Altschuler has begun to investigate the problem of curve shortening for space curves focusing on the types of singularities that may develop[3]. The current paper uses these results to investigate the relationship between the curve shortening flow (CSF) and knot theory with particular emphasis on the curve's writhe (Wr) and the link (Lk) and twist (Tw) of the curve's Frenet ribbon.

The project was done in several stages, which are presented in roughly chronological order in this paper. First, the necessary background in differential geometry of curves had to be acquired (§2), then the background in knot theory (§3), and then the curve shortening evolution (§4) which contains new results. Section 5 contains numerical examples, while most conjectures and suggestions for further research are in §6. Sections 2 and 4 are related but do not directly build off of each other and both are distinctly separate from §3. Sections 2 and 3 provide most of the notation for the paper. Also there were a number of things studied and interesting things learned that have been omitted for the sake of clarity, focus and brevity.

I would like to thank Prof. Martin Jackson for introducing me to differential geometry and the curve shortening problem and especially for supervising this research project. Thanks are also due to Prof. Bryan Smith for his helpful conversations concerning knot theory, the UPS Mathematics department for their support and interest, and the Adam S Goodman scholarship and the University of Puget Sound for funding this project. Additionally all Surface Evolver datafiles and *Mathematica*® notebooks are available from the author via email request to jpreszler@member.ams.org.

2. DIFFERENTIAL GEOMETRY OF CURVES

For the remainder of the paper let $r : I_1 \times I_2 \rightarrow \mathbb{R}^n$ be a family of parameterized curves, each member having a fixed value of the evolution parameter. The parameter u shall be used for an arbitrary spatial parameter, s for the arc length parameter, and t for the evolution parameter. We will further restrict ourselves to the cases $n = 2$ and $n = 3$ and if the distinction between the two possibilities is important then the curve will be specified as a plane or space curve, respectively. Also let $r' = \frac{\partial r}{\partial u}$ unless otherwise noted. The notation $r(u, \cdot)$ or simply $r(u)$ will be used where ever the evolution parameter t is of no importance.

Throughout this paper we will require the existence of the Frenet frame defined by

$$(1) \quad \frac{\partial}{\partial s} \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Here k is the curvature, τ is the torsion (both are functions of arc length), and T , N , and B are the unit tangent, normal, and binormal vectors respectively. Also the unit binormal is defined as $B = T \times N$. The Frenet frame requires that we have a *regular* curve, i.e. $r'(u) \neq 0 \forall u \in I_1$, and that the $k(s) \neq 0 \forall s \in I_1$.

There are more useful computational formulas for the curvature and torsion that depend only on the initial curve $r(u, \cdot)$. These formulas are:

$$(2) \quad k(u) = \frac{|r'(u) \times r''(u)|}{|r'(u)|^3}, \text{ and}$$

$$(3) \quad \tau(u) = \frac{(r'(u) \times r''(u)) \cdot r'''(u)}{|r'(u) \times r''(u)|^2}.$$

If $r(u)$ is a planar curve then the torsion is always zero and it is possible to put a sign on the curvature; see [7] for more on this subject since our primary focus is space curves.

The existence of the Frenet frame allows us to discuss the so-called Fundamental Theorem of the Local Theory of Curves.[7]

Theorem 1 (Fundamental Theorem of the Local Theory of Curves). *Given differentiable functions $k(s) > 0$ and $\tau(s)$ for $s \in I_1$, there exists a regular parameterized curve $r : I_1 \rightarrow \mathbb{R}^3$ such that s is the arc-length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of $r(s)$. Furthermore, any other curve, $\bar{r}(s)$, with the same arc-length, curvature, and torsion, differs from $r(s)$ by an orientation-preserving rigid motion. If the arc-length and curvature are the same but the torsion of $\bar{r}(s)$ is the negative torsion of $r(s)$ then the two curves differ by an orientation-reversing rigid motion.*

Thus, to uniquely determine a curve in three-space we only need an initial point on the curve and the curvature and torsion functions. This is a powerful result and clearly demonstrates the importance of the curvature and torsion. Thus, the behavior of these two functions will be of the utmost importance when studying the curve shortening flow. A proof

of the uniqueness part of this theorem requires little more than the Frenet equations (1), but the existence uses ideas from ODE's. Proofs of both can be found in [7] and elsewhere.

While the fundamental theorem tells us that all we need is curvature and torsion, the local canonical form helps us construct the curve. Also, the local canonical form provides a natural representation of a curve with respect to the coordinate system of the Frenet frame. This form is determined by using the finite Taylor expansion

$$r(s) = r(s_0) + s * r'(s_0) + \frac{s^2}{2} * r''(s_0) + \frac{s^3}{6} * r'''(s_0) + R,$$

where R is the remainder term satisfying $\lim_{s \rightarrow s_0} \frac{R}{s^3} = 0$. Using (1) and the coordinate system $(x, y, z) = (T(s), N(s), B(s))$ with $r(s_0)$ being the origin, we can define $r(s) = (x(s), y(s), z(s))$ by

$$(4) \quad x(s) = s - \frac{k^2 s^3}{6} + R_x,$$

$$(5) \quad y(s) = \frac{ks^2}{2} + \frac{k's^3}{6} + R_y,$$

$$(6) \quad z(s) = \frac{k\tau s^3}{6} + R_z,$$

where k' is differentiation with respect to arc-length. We can now start at any point on the curve and proceed to construct the entire curve knowing only how much the curve bends (curvature) and twists (torsion) from point to point.¹

So far we have concerned ourselves with the local theory of curves and have yet to look at more global structure. The global properties of curves will be of more importance when examining the CSF than the local properties. However, very little global theory can be developed without a specific family of curves so only one of the simpler, yet useful, facts from the general global theory of curves will be discussed immediately.

The isoperimetric problem is one of the oldest in differential geometry, being known and solved (in the planar case) by the ancient Greeks. The problem asks: "Of all simple closed curves in the plane with given length L , which one bounds the largest area?" [7] Here a simple curve is one that doesn't intersect itself except at its endpoints (since it's closed). This problem is commonly seen in high school geometry classes, with the circle of circumference L being the solution. However, this problem wasn't rigorously proved because a solution was assumed to exist. In 1870 Weierstraß pointed out that similar problems fail to have a solution and gave the first rigorous proof of the isoperimetric inequality. His proof was very difficult and has since been simplified, but not enough for inclusion here. We shall state the isoperimetric inequality and refer the interested reader to [7] for the proof.

¹We will note that the torsion, measuring the twisting of the curve, is indeed related to the topological twist (Tw) of a ribbon. This relationship will be made explicit in §3.

Theorem 2 (Isoperimetric Inequality). *Let X be a simple closed curve in the plane with length L and bounding a region of area A . Then*

$$(7) \quad L^2 - 4\pi A \geq 0,$$

and equality holds iff X is a circle.

In addition to the plane, finding isoperimetric inequalities for curves on other surfaces has been an interesting problem in differential geometry. The interested reader is referred to [20] for the case of a sphere. There are also results concerning isovolumetric inequalities and similar properties for higher dimensions. One interesting feature of the isoperimetric inequality is that it provides an easy way of determining the existence of a curve with certain hypothetical quantities. For example we know that a curve with length of 6 units, bounding an area of 3 square units, can't exist.

The isoperimetric inequality has also demonstrated its utility in the investigation of the CSF. Gage [11] proved that a simple, closed, convex, planar curve will become a circle under the CSF. His proof consists of showing that the length and area evolve in such a way that the isoperimetric inequality holds throughout the deformation and that equality holds in the limit of the evolution process.

3. KNOT THEORY

Knot theory is one of the few areas of mathematics that involves the study of exactly what the name implies to a layperson, in this case knots. However, these knots are formed by creating a knot in a piece of string and then attaching the ends of the string together, thus making it impossible to untie unless your original knot wasn't really a knot (such a knot is called the *unknot*). Much of the following can be found in [1], as well as a great deal more; this is a highly recommended book for those interested in knot theory.

Knot theory originated in the 1880's when Lord Kelvin came up with the idea that atoms were simply knots in "ether." The Scottish physicist Peter Tait, now known for being one of the first knot theorists, began to tabulate different types of knots hoping that he could delineate all possible elements. C.N. Little, an American mathematician of the same period, also began producing tables of knots. As we all know Kelvin's ideas concerning knots in ether were wrong, but the mathematicians had found an interesting new subject and continued their study despite the immediate lack of application.

However, in the 1980's biologists and chemists discovered that certain molecules like DNA knotted themselves up. How this coiling occurred became an important question, to which topology and knot theory quickly became of use. Also chemists began to produce knotted molecules synthetically, where the type of knot determined the molecular properties. Much of the knot theory of importance to this paper is related to the super-coiling of DNA.

Since a true knot can never be untied (in knot theory at least) it is an interesting problem to figure how to tell if two knots are indeed the same, thus enabling one to tell if a knot

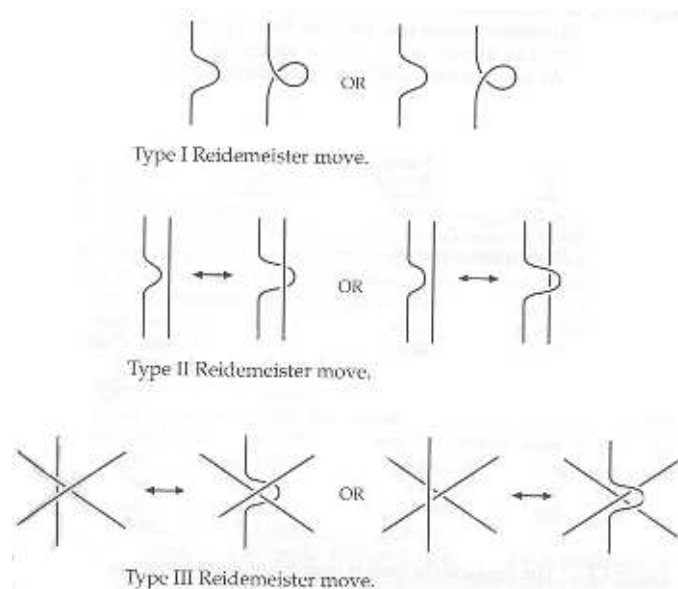


FIGURE 1. Reidemeister moves of type I, II, and III. Image taken from [1]

is the *unknot*. Since knots can be deformed physically without changing any of their true structure this problem is quite difficult and there are many unanswered questions relating to knot isomorphism, but many important quantities have been developed to help categorize knots. The quantities of link, twist, and writhe will be explained after the discussion of Reidemeister moves and the construction of the Frenet ribbon.

A curve alone can be knotted, and some knot quantities make sense for single closed curves such as writhe which depends only on the knot's axis curve. However most quantities, such as link and twist, require a tube or ribbon. A Frenet ribbon can be constructed from the base curve by moving some $\epsilon > 0$ distance out along the Frenet normal vector. If $r(u)$ is the base curve then the other edge of the Frenet ribbon is given by $r(u) + \epsilon * N(u)$ and the ribbon itself is a flat surface stretched between the two curves. These two curves will be denoted as $r(u)$ for the base curve and $\bar{r}(u)$ for the Frenet ribbon curve.

Let us now look at one method of determining if two knots are isomorphic, namely Reidemeister moves. In 1926 Kurt Reidemeister, a German mathematician, proved that two knots are the same if you could use a sequence of "Reidemeister moves" to get from one to the other. There are three types of Reidemeister moves and all of them change the crossings that the knot makes with itself, but they change these crossings in a "trivial" way by not allowing part of the knot to be passed through itself and thus preserve the structure. Type I moves allow us to add or remove a loop in the knot. Type II moves allow us to add or remove two crossings of different sign and type III moves enable us to move a strand of the knot from one side of a crossing to the other. These moves are depicted in figure 1.

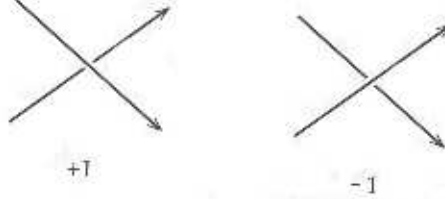


FIGURE 2. Signed crossing convention. Image from [1]

The linking number of a Frenet ribbon is one half the sum of its signed crossings, but can also be defined analytically as

$$(8) \quad Lk(r, \bar{r}) = \frac{1}{4\pi} \int \int \frac{(T_r(v) \times T_{\bar{r}}(s)) \cdot (r(v) - \bar{r}(s))}{|r(v) - \bar{r}(s)|^3} dv ds,$$

where both s and v are arc-length parameters and $\bar{r} = r + \epsilon N$. Clearly the method of adding the signed crossings is much easier because we only have to look at one projection of the knot (Reidemeister moves can get us to any other projection, but they don't change Lk). Crossings are given a sign of ± 1 according to figure 2.

The linking number has several interesting properties. One is that it's invariant under Reidemeister moves, which is a consequence of being a topological invariant. Also $Lk \in \mathbb{Z}$ and Lk changes by ± 1 if the ribbon is passed through itself. Additionally, Lk is the only knot theoretic quantity considered here that is easy to see exactly from simply looking at the knot; both twist and writhe require a high degree of computation.

The twist of a Frenet ribbon measures the rate at which the ribbon curve winds around the base curve, or more specifically how the normal vector winds around the tangent vector in the Frenet frame. Thus the total twist (Tw) is related to the total torsion of the base curve by

$$(9) \quad Tw(r(s), N(s)) = \frac{1}{2\pi} \int \tau(s) ds.$$

An important thing to note is that the above formula is specific to the Frenet ribbon. The twist is an additive quantity, thus the twist of two composed knots is the twist of the first plus the twist of the second.

Writhe is the most abstract quantity that will be dealt with, and has several interpretations. The two with the most intuitive importance are that it is the difference between the link and twist (see White's formula below), but writhe also has an interesting area interpretation. This interpretation was first mentioned by Fuller [10] in 1978 and latter justified by Cantarella [6]. This area interpretation says that the writhe of a simple closed curve will

satisfy

$$(10) \quad 1 + Wr \equiv \frac{A}{2\pi} \pmod{2},$$

where A is the area on the unit sphere enclosed by the tangent indicatrix of the curve. The tangent indicatrix is a curve on the unit sphere formed from the endpoints of the original curve's unit tangent vectors based at the origin of the sphere. From this formula we can see that the writhe of a planar curve must be a multiple of 2, thus demonstrating the short-coming in this interpretation. Ideally, we would know the exact multiple of two, such knowledge requires the use of a different interpretation. Since a planar curve is unlinked with it's ribbon and the torsion is everywhere zero, by White's formula (12) the writhe should be zero. This can be seen from the analytical definition of writhe:

$$(11) \quad Wr(r) = \frac{1}{4\pi} \int \int \frac{(T(v) \times T(s)) \cdot (r(v) - r(s))}{|r(v) - r(s)|^3} dv ds,$$

where both s and v are arc-length parameters.²

From the analytic definition we can see that a planar curve indeed has zero writhe because the cross product of the tangents produces a vector normal to the plane of the curve. Since this vector is orthogonal to $r(v) - r(s)$ (which is in the plane of the curve) the numerator of the integrand is always zero.

Another interpretation of writhe comes from the clear similarity between equations (11) and (8). The writhe is also referred to as the "self-linking number" for this reason. However the writhe is not a topological invariant so we can obtain it by counting the signed crossings, but we must do so over every possible projection which make the integral calculation more appealing. Neither writhe nor twist must be integers, but their sum must be as a consequence of White's formula below.

Link, twist, and writhe are related by

$$(12) \quad Lk = Tw + Wr.$$

This is commonly called White's formula and explains why the link and twist are useful quantities to consider when studying the writhe.

Since Reidemeister moves change the relationship of crossings (or the number of crossings) some Reidemeister moves can affect knot quantities such as the link, twist, and writhe. It can easily be seen that the linking number, or link, is unchanged under Reidemeister move.

Note that a Reidemeister move of type I will change the total twist in a knot. However the linking number must remain constant so by White's formula the writhe must also change under a type I move. This is clear evidence that neither twist nor writhe are topological

²The literature is somewhat vague concerning this formula. Some articles make it seem that any spatial parameterization can be used, while others hint at only an arc-length parameterization. From the numerous numerical examples worked out during my research the curve must be parameterized by arc-length and the integration must be done with respect to arc-length.

invariants, but they are both invariant under type II and III moves. In fact the twist and writhe are invariant under dialations and contractions of space and orientation-preserving rigid motions. Only the sign of link, twist, and writhe are changed under orientation-reversing rigid motions.

Lastly it is mentioned in [2] and elsewhere that writhe jumps by $\pm 1^3$ if the curve is deformed through itself and the twist changes continuously even if the ribbon is deformed through itself. However, no explanations of this result are given and some of the numerical evidence obtained during this research project seem to conflict with the claims in [2]. This will be explained fully in §5 and 6.

4. CURVE SHORTENING EVOLUTION

Let $r(u, t)$ be a simple, closed curve. The curve shortening flow moves every point on r in the direction of its Frenet normal vector at a rate equal to the curvature at that point. This can be described by

$$(13) \quad \frac{\partial r}{\partial t} = kN.$$

It is useful to have a relationship between differentiation with respect to arc-length and with respect to a general space parameter, this is given throughout the literature as

$$(14) \quad \frac{\partial}{\partial s} = \frac{1}{q} \frac{\partial}{\partial u},$$

where $q = |\frac{\partial r}{\partial u}|$. Also we can commute $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ according to

$$(15) \quad \frac{\partial}{\partial t} \frac{\partial}{\partial s} = \frac{\partial}{\partial s} \frac{\partial}{\partial t} + k^2 \frac{\partial}{\partial s}.$$

It is interesting to note that these relationships hold for both planar and space curves.

Before proceeding with the evolution equations for spatial quantities we will mention some of the important results concerning curves in the plane. In general, under the CSF a simple, closed, convex, planar curve becomes circular as $t \rightarrow \omega$ where ω is the time where equality holds in the isoperimetric inequality. If the curves are initially embedded they remain so under the evolution and the only singularities one must be concerned with are self-intersections of the curve. The work of many mathematicians went into developing these, and other, results and is still an ongoing effort. Those interested in the planar case are referred to [11], [12], [14], and [8]. Steven Altschuler together with his advisor Matt Grayson have recently began to explore the CSF for space curves, [3] [4]. Their work originated with trying to resolve planar singularities through the use of “ramps” in three-space (see [4]), but the basic results hold for arbitrary space curves.

Let us now look at how particular quantities of the curve evolve under the CSF, in particular the tangent vector, curvature, and torsion. The tangent vector evolves according

³In [2] ± 2 is used, but this is because of a different method of adding the signed crossings.

to

$$(16) \quad \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial s^2} + k^2 T.$$

This form is found in [4], but can also be expressed as,

$$(17) \quad \frac{\partial T}{\partial t} = \frac{\partial k}{\partial s} N + k \tau B,$$

which is more closely related to the planar evolution of the tangent.⁴

The evolution equations for curvature and torsion are given in [4], and repeated here for reference, as

$$(18) \quad \frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k(k^2 - \tau^2)$$

and

$$(19) \quad \frac{\partial \tau}{\partial t} = \frac{\partial^2 \tau}{\partial s^2} + \frac{2}{k} \frac{\partial k}{\partial s} \frac{\partial \tau}{\partial s} + \frac{2\tau}{k} \left(\frac{\partial^2 k}{\partial s^2} - \frac{1}{k} \left(\frac{\partial k}{\partial s} \right)^2 + k^3 \right).$$

From the evolution equations for the curvature and torsion one can see that even if the curvature is bounded, the torsion can “blow-up,” because the curvature can be arbitrarily close to 0 and thus $\frac{1}{k}$ becomes arbitrarily large.

One with a background in PDE’s will notice that so far all quantities seem to evolve according to heat equations, or parabolic PDE’s. This demonstrates one main aspect of the CSF, it minimizes the curves “energy,” seeking to smooth out the curve. Heat equations have the characteristic of propagating information at infinite speeds. Since the above equations have lower order terms we don’t see things evolving at infinite speed but these quantities do smooth out fairly quickly. There are some quantities, most notably writhe and length that don’t evolve according to heat equations, so one doesn’t see the length minimizing at infinite speeds, but instead shows an almost linear decrease in most examples.

The length of a curve, L , evolves by

$$(20) \quad \frac{\partial L}{\partial t} = - \int k^2 ds.$$

Later we will see that this is closely related to the evolution of the writhe.

A result from differential geometry is that a helix has a constant ratio between the torsion and the curvature, or

$$\frac{\tau}{k} = \text{constant}.$$

Thus it is natural to ask if a helix will remain a helix during the CSF? The following theorem shows that this ratio doesn’t necessarily remain constant.

⁴In [12] the evolution for the tangent vector of a planar curve is

$$\frac{\partial T}{\partial t} = \frac{\partial k}{\partial s} N.$$

Equation (17) is equivalent to this since the torsion of a plane curve is zero.

Theorem 3. *Let $k(s) \neq 0 \forall s \in I_1$ be the curvature and $\tau(s)$ be the torsion of a simple space curve $r(s, t)$ where s is arc-length and t is the deformation parameter. Then $\frac{\tau}{k}$ evolves according to*

$$(21) \quad \frac{\partial}{\partial t} \left(\frac{\tau}{k} \right) = \frac{\partial^2}{\partial s^2} \left(\frac{\tau}{k} \right) + \frac{4}{k^2} \frac{\partial k}{\partial s} \frac{\partial}{\partial s} \left(\frac{\tau}{k} \right) + \frac{2\tau}{k^3} \frac{\partial^2 k}{\partial s^2} + \tau k + \frac{\tau^3}{k}.$$

Proof. This evolution equation follows immediately from the quotient rule with substitutions using equations (18), (19), and the quotient rule for $\frac{\partial^n}{\partial s^n} \left(\frac{\tau}{k} \right)$. \square

Now we will derive the evolution equation for writhe, beginning from a known result. From [2] we have the following theorem.

Theorem 4. *Let X be a simple closed space curve with deformation parameter $\lambda \in [0, 1]$, and tangent vectors given by $T(s, \lambda)$ where s is the arc-length parameter. If $T(s, 0) \neq -T(s, \lambda) \forall \lambda \in [0, 1]$ then*

$$(22) \quad \frac{d}{d\lambda} Wr(X_\lambda) = -\frac{1}{2\pi} \int \left(\frac{\partial}{\partial \lambda} T(s, \lambda) \times T(s, \lambda) \right) \cdot \frac{\partial}{\partial s} T(s, \lambda) ds.$$

Note that the condition $T(s, 0) \neq -T(s, \lambda)$ is equivalent to saying that there are no discontinuities in the normal vector field during evolution.

By translating the above theorem into our notation and simplifying by (1) we obtain the following equivalent theorem, which is our main result.

Main Theorem. *[Evolution of Writhe] Let $r(s, t) : I_1 \times I_2 \rightarrow \mathbb{R}^3$ be a simple, closed space curve with arc-length parameterization s and evolution parameter t . Let k be the curvature function and τ the torsion function of r . If $T(s, 0) \neq -T(s, \lambda) \forall \lambda \in [0, 1]$ then*

$$(23) \quad \frac{\partial}{\partial t} Wr(r) = -\frac{1}{2\pi} \int k^2 \tau ds.$$

Proof. By changing notation in (22) we obtain

$$(24) \quad \frac{\partial}{\partial t} Wr(r) = -\frac{1}{2\pi} \int \left(\frac{\partial}{\partial t} T(s, t) \times T(s, t) \right) \cdot \frac{\partial}{\partial s} T(s, t) ds.$$

By (1), the evolution of the tangent (16), and the properties of the cross and dot products of vectors in the Frenet frame we obtain

$$(25) \quad \frac{\partial}{\partial t} Wr(r) = -\frac{1}{2\pi} \int \left(-\frac{\partial k}{\partial s} k(B \cdot N) + k^2 \tau(N \cdot N) \right) ds.$$

Since $B \cdot N = 0$ and $N \cdot N = 1$, (25) reduces to (23). \square

Since we now know the formula that dictates the change in writhe under the CSF, it is natural to ask how the change in writhe evolves under the curve shortening flow, i.e. what is $\frac{\partial^2}{\partial t^2} Wr$?

Theorem 5. *[Evolution of $\frac{\partial}{\partial t}Wr$] Let $r(s, t) : I_1 \times I_2 \rightarrow \mathbb{R}^3$ be a simple, closed space curve with arc-length parameterization s and evolution parameter t . If k is the curvature function and τ the torsion function of r , then*

$$(26) \quad \frac{\partial^2}{\partial t^2}Wr(t) = -\frac{1}{\pi} \int k\tau \left(2\frac{\partial^2 k}{\partial s^2} + \frac{\partial k}{\partial t} \right) ds.$$

Proof. By (14) and (23) we must compute

$$(27) \quad \frac{\partial^2}{\partial t^2}Wr(r) = -\frac{1}{2\pi} \int \frac{\partial}{\partial t}(k^2\tau) du.$$

From the product rule, simplification, and integration by parts (27) becomes

$$(28) \quad \frac{\partial^2}{\partial t^2}Wr(r) = -\frac{1}{2\pi} \int \left(\frac{\partial^2}{\partial s^2}(k^2\tau) + 6k\tau \frac{\partial^2 k}{\partial s^2} + 2k^2\tau(k^2 - \tau^2) \right) ds.$$

By the additivity of the integral we can remove the first term, which is zero because of the periodicity of $k^2\tau$, and a factor of 2 can be removed from the remainder which yields

$$(29) \quad \frac{\partial^2}{\partial t^2}Wr(r) = -\frac{1}{\pi} \int \left(3k\tau \frac{\partial^2 k}{\partial s^2} + k^2\tau(k^2 - \tau^2) \right) ds.$$

While this may be the most useful form (since it is entirely in terms of curvature, torsion, and partials with respect to arc-length) we note that a further substitution from (18) produces (26). \square

5. NUMERICAL EVIDENCE

Space curves with desired properties that are easy to work with are notoriously difficult objects to find in general and examples of simple, closed non-trivial space curves are certainly no exception. Fortunately from [22] we have an interesting family of simple, closed space curves given by $S_n : [0, 2n\pi] \rightarrow \mathbb{R}^3$ defined as

$$(30) \quad S_n(u) = \left(\left(1 - \frac{\cos(u)}{n} \right) \cos\left(\frac{u}{n}\right), \left(1 - \frac{\cos(u)}{n} \right) \sin\left(\frac{u}{n}\right), \frac{\sin(u)}{n} \right).$$

In [22] it is shown that $\frac{\tau}{k}$ approaches a non-zero constant as $n \rightarrow \infty$; however we will only concern ourselves with small values of n .

In addition to the above family of space curves, several knot parameterizations were also used extensively, such as

$$(31) \quad g(u) = (3\sin(2u) - \sin(u), \sin(3u), 3\cos(2u) + \cos(u)),$$

and a serendipitously derived variant that has some interesting behavior

$$(32) \quad h(u) = (-3\sin(2u) - \sin(u), -\sin(3u), 3\cos(2u) + \cos(u)).$$

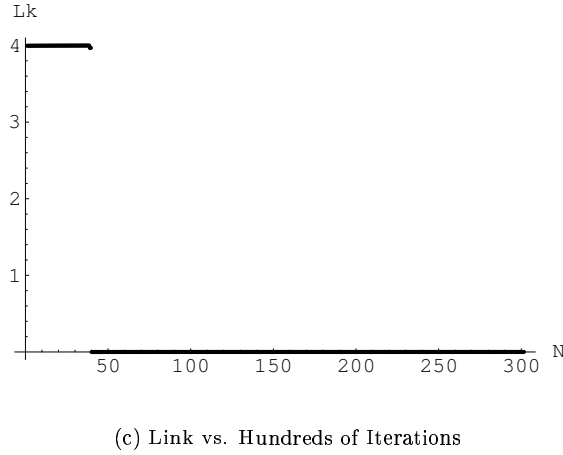
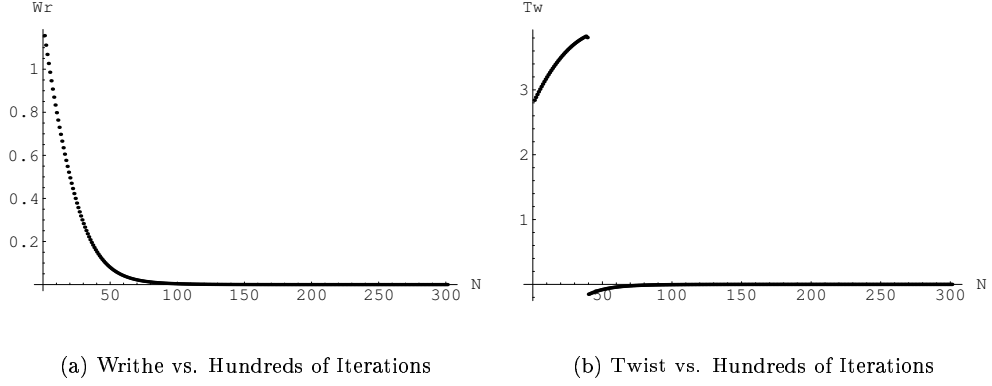


FIGURE 3. Evolution of the writhe, twist, and link of (30) when $n = 4$.

While these are all nice curves, they don't evolve by themselves. To evolve the curves Ken Brakke's *Surface Evolver* software was used and can be freely downloaded from his web-site (<http://www.susqu.edu/facstaff/b/brakke/evolver/>). This program reads datafiles containing the definition of the curve and some initial data and commands. The curve can then be refined (producing a better approximation) and evolved according to the CSF (or other types of flows). Technically Surface Evolver seeks to minimize the user specified energy of the structure, which in our case is the arc-length of the curve.

Many useful quantities are built into Evolver and quantities not built in can be programmed into Evolver by the user. This allows us to capture data on curvature, torsion, writhe, twist, and the length of the curve during the evolution process. The captured data can then be plotted using *Mathematica*®.

Figures 3 - 4 are graphs of the evolution of the knot theoretic quantities for $S_4(u, t)$ (eq. 30) and $h(u, t)$ (eq. 32), where the y -axis is the value of the quantity and the x -axis is

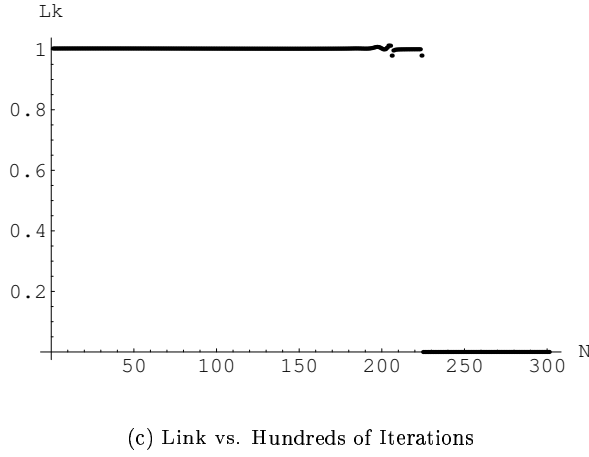
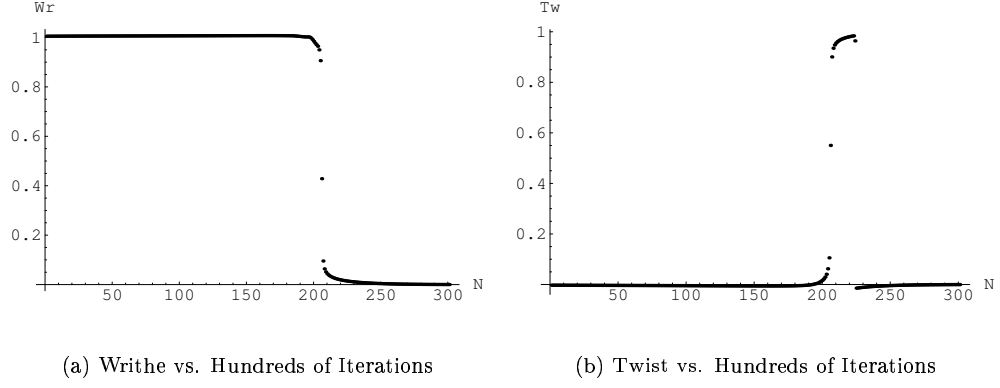


FIGURE 4. Evolution of the writhe, twist, and link of (32).

the number of iterations of the evolution process (1 iteration corresponds to roughly .001 seconds).

These two curves behave very differently under the evolution process. For $S_4(u, t)$ (eq. 30) the curve and its ribbon never intersect with themselves during the evolution process. The only interesting points are inflection points, where $k = 0$, which are the locations of crossings between the edges of the ribbon. These crossings appear in a flat projection of the ribbon and are used to compute the linking number.

For $h(u, t)$ (eq. 32) the curve and the ribbon have a self-intersection after approximately 220 iterations. Evolver handles this very nicely and leads one to question whether this is the only “nice” way of handling a self-intersection.

6. CONCLUSIONS

However, both of the above curves suggest several common events. One, mentioned in §3, is the apparent contradiction between these numerical examples and the claims in [2] that twist changes continuously during deformation and writhe will experience a jump if the ribbon is passed through itself. Resolving this is of the utmost importance before the current lines of research can be continued. The discontinuity in the evolution of the twist leads to the first of several conjectures.

Conjecture 1 (Frenet Unlinking). If the Frenet ribbon of a closed space curve is initially linked (i.e. $Lk \neq 0$) then the curve shortening flow will unlink ($Lk = 0$) the ribbon in finite time.

Additionally all numerical evidence to date points to the following conjectures concerning the behavior of the writhe under the CSF. For the following let Wr_0 denote the initial writhe of a curve and let Wr_t denote the writhe of the curve at time $t \geq 0$ of the evolution process.

Conjecture 2 (Writhe behavior). If the writhe of a closed space curve is defined during the curve shortening evolution on the interval $[0, \omega)$ for some real number $\omega > 0$, then

$$(33) \quad |Wr_t| \leq |Wr_0| \quad \forall t \in [0, \omega).$$

Furthermore, if $Wr_0 \geq 0$ then $Wr_t \geq 0$ and if $Wr_0 \leq 0$ then $Wr_t \leq 0 \quad \forall t \in [0, \omega)$.

From the previous conjecture it should be easy to establish the stronger result below.

Conjecture 3 (Monotonicity of Writhe). If $Wr_0 \geq 0$ and the CSF is defined on the interval $[0, \omega)$ then $Wr_t \leq Wr_p$ where $0 \leq p < t < \omega$. Similarly, if $Wr_0 \leq 0$ then $Wr_t \geq wr_p$ where $0 \leq p < t < \omega$.

Since a planar curve must have $Wr_0 = 0$, if the curve is embedded then $Wr_t = 0 \quad \forall t$. However, $Wr_t = 0$ does not imply that the curve is planar at time t since a spherical curve also has $Wr = 0$. This hints at a possible classification problem for curves based on their writhe, which would be a purely topological problem.

Conjecture 4 (Classification of Curves by Writhe). If the writhe of a simple closed curve is integral, $|Wr| = n \in \mathbb{N}$, then n is the minimum genus of the surface that the curve can be embedded in. If $Wr = 0$ then the curve is either planar or spherical.

A good initial step towards the resolution of the classification conjecture would be to derive a new formula for the writhe exclusively in terms of the curvature and torsion of the curve.

Lastly we would like to know the following:

Conjecture 5. If

$$\frac{\partial^n}{\partial t^n} Wr = 0$$

then

$$\frac{\partial^{n+1}}{\partial t^{n+1}} Wr = 0$$

for $n \geq 0$.

This is intimately related to the previous conjectures concerning the behavior of the writhe during the evolution process which is part of the overall question:

Conjecture 6. If a closed space curve where $Wr_t = 0$ for some $t \in [0, \omega)$ then $Wr_p = 0$ $\forall t \leq p \leq \omega$.

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